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$1, 4, 3, -1, -4, -3\,;\ \dot{1}, \dot{4}, \dot{3}, -\dot{1}, -\dot{4}, -\dot{3}$, show that, if the sum of these products be divisible by 13, so N will be, and not otherwise." The reason for the rule is apparent when we notice that $1, 4, 3, -1, -4, -3$ are the remainders in reverse order of $10^1, 10^2, 10^3, 10^4, 10^5, 10^6$ mod. 13 ; or what is the same thing in the development of $\frac{1}{13}$ as a circulating decimal.

Since we may prefix any number of ciphers to any number, it is clear that we may start with any number of the series only being careful to preserve the cyclical order. For example, we might equally as well write the series $3, -1, -4, -3, 1, 4$.

Example. 11140640173 is divisible by 13 because $1(1)+4(1)+3(1)-1(4)-4(0)-3(6)+1(4)+4(0)+3(1)-1(7)-4(3)=-26=-2(13)$.

728 is divisible by (13) because $3(7)-1(2)-4(8)=-13$.

The reason for the rule suggests its extension to any number whatever.

Thus $\frac{1}{3}$ developed in a circulating decimal gives the constant remainder 1 and we have the well known rule that a number is divisible by 3 if the sum of its digits is so. $\frac{1}{7}$ developed in a circulating decimal gives the series $2, 3, 1, -2, -3, -1$. Thus 6028620892 is divisible by 7 because $2(6)+3(0)+1(2)-2(8)-3(6)-1(2)+2(0)+3(8)+1(9)-2(2)=7$.

For 11 the remainders are $1, -1$, and we have the known rule for divisibility by 11. For 13 the rule is as stated by Sylvester. For 17 we find the series $1, -5, 8, -6, -4, 3, 2, 7, -1, 5, -8, 6, 4, -3, -2, -7$. Thus 442 is divisible by 17 because $3(4)+2(4)+7(2)=34=2(17)$.

For 19 we have the series $1, 2, 4, 8, -3, -6, 7, -5, 9, -1, -2, -4, -8, 3, 6, -7, 5, -9$. It is clear that in this way we can find similar tests of divisibility for any number whatever, but it does not seem worth while to push the matter further except in special cases.

A simple rule for divisibility by 37 may be found in this way. The remainders are $1, -11, 10$. Thus 343619 is divisible by 37 because $1(3)-11(4)+10(3)+1(6)-11(1)+10(9)=74=2(37)$.

May 7, 1897.

INTRODUCTION TO DIFFERENTIATION.

By JOHN MACNIE, A. M., Professor of Mathematics, University of North Dakota.

1. In the identity $\dfrac{r^n-1}{r-1}=r^{n-1}+r^{n-2}+\ldots\ldots r+1, \quad \ldots\ldots\ldots\ldots\ldots(1),$

Since r may have any value, let $r=\dfrac{x^{1/m}}{z^{1/m}}$; then, by substituting this value for r in (1), multiplying both members by $z^{\frac{n-1}{m}}$, and simplifying, we obtain

$$\frac{x^{n/m}-z^{n/m}}{x^{1/m}-z^{1/m}} = x^{\frac{n-1}{m}} + x^{\frac{n-2}{m}}z^{\frac{1}{m}} + \ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}} + z^{\frac{n-1}{m}} \ldots\ldots\ldots (2).$$

Dividing both members of (2) by the factor that rendered $x^{1/m}-z^{1/m}$ rational, we obtain, since, by (2), $x-z=(x^{1/m}-z^{1/m})(x^{\frac{n-1}{m}}+\ldots\ldots z^{\frac{n-1}{m}})$,

$$\frac{x^{n/m}-z^{n/m}}{x-z} = \frac{x^{\frac{n-1}{m}}+z^{\frac{n-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{n-2}{m}}+z^{\frac{n-1}{m}}}{x^{\frac{m-1}{m}}+x^{\frac{m-2}{m}}z^{\frac{1}{m}}+\ldots\ldots x^{\frac{1}{m}}z^{\frac{m-2}{m}}+z^{\frac{n-1}{m}}} \ldots\ldots\ldots (3),$$

which, as m may have any value, ± 1 included, is a general expression for the *ratio of the difference of two like powers to the difference of their bases.*

In (3), if we suppose $z=x$, since, then, there are in the numerator of the second member n terms, each$=x^{\frac{n-1}{m}}$, and in the denominator m terms, each$=x^{\frac{m-1}{m}}$ we obtain, for $z=x$,

$$\left[\frac{x^{\frac{n}{m}}-z^{\frac{n}{m}}}{x-z}\right]_{x=z} = \frac{0}{0} = \frac{nx^{\frac{n-1}{m}}}{mx^{\frac{m-1}{m}}} = \frac{n}{m}x^{(n/m)-1}$$

the first member assuming the indeterminate form on account of the presence in numerator and denominator of the factor $x^{1/m}-z^{1/m}$, which becomes zero by hypothesis. Hence, as m may have any value, the formula

$$\left[\frac{x^n-z^n}{x-z}\right]_{x=z} = nx^{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

holds true for every value of n. For the sake of simplicity of statement we shall suppose in what immediately follows $m=1$, and n=a positive integer.

Then (3) becomes

$$\frac{x^n-z^n}{x-z} = x^{n-1}+x^{n-2}z+\ldots\ldots xz^{n-2}+z^{n-1} \quad \ldots\ldots\ldots\ldots (3').$$

2. Now, instead of regarding x and z in (3′) as unknown constants, we may regard them as denoting different values of the same variable z, as it varies from $z=0$, through $z=x$, toward $z=+\infty$. From this point of view we see that, assigning any two values to x and z, each member of (3′) expresses *the ratio of the increment of the power to the increment of the base, between these values;* or, briefly expressed, gives the *rate of increase of z^n.* For example, let $z=0$, $x=a$; then both members of (3′) become a^{n-1}, the *average* rate of increase of z^n while z increases from 0 to a; i. e. while z has increased by a units, z^n has increased a^{n-1} times as fast. We say "average rate" because, as will be seen by giving different values to a, the rate of increase of z^n is continually accel-

erating, just as the velocity or rate of motion of a falling body is continually accelerating.

3. If now we suppose $z=x-h$, h being infinitely small, the second member of (3′) will be *less* than nx^{n-1} by a difference infinitely small ; and if we suppose $z=x+h$, the second member of (3′) will be *greater* than nx^{n-1} by a difference infinitely small ; we infer, accordingly (the values of that second member being continuous) that nx^{n-1} represents the *rate of increase of* z^n *when* z *is passing through* x. For, if nx^{n-1} does not represent the rate of increase of z^n when z is passing through x, for what value of z does it represent the rate ?

The difficulty that is here experienced arises from the fact that we have here to deal, not with a constant ratio, as in algebra, but with a ratio that varies continuously as its terms vary, ratios of frequent occurence in physics and kindred sciences. Thus, when we say that a falling body at a certain point in its descent has a velocity of 50 feet a second, we do not mean that the body moves at that rate during any assignable period of time, but *would* descend that distance in a second, *if the motion continued uniform.* In the same way, nx^{n-1} does not mean the rate of increase of z^n during an interval of increase of z but the rate at which z^n would increase if the rate became constant from x.

From the limitation of our faculties, we are unable to realize the *absolute ;* as, for example, to draw or even conceive a straight line absolutely without breadth. Yet, while admitting this inability, we ignore in our reasonings upon straight lines all that is inconsistent with their definition. Similarly, while in our conception of a variable, a changing velocity for example, we can not help thinking of the *element* of change as constant for some interval, however minute, we here, again, ignore whatever is inconsistent with the definition of a variable as changing continuously. There is no objection, then, to our assisting our grasp of the idea by regarding a power of a variable as changing by infinitely small constant* *elements*, as long as we ignore inconsistent consequences.

4. DEF. *Function*, as usual. Example, x^n a function of x.

5. DEF. A variable being supposed to change by infinitely small *elements*, such an element is called the *differential* of the variable. The differential of a variable is denoted by the symbol d prefixed to the symbol of the variable. Thus dx, $d(x^n)$, are read respectively, *the differential of* x, *the differential of* x^n.

It has already been seen that $d(x^n)=nx^{n-1}dx$, that is when the variable is passing through the value x, the power is changing nx^{n-1} times as fast as the variable. Hence nx^{n-1} is called *the differential coefficient* of x^n, etc.

6. (Here would follow the demonstration of the rules for algebraic sums of variables, found much as usual. The rule for the differential of products may be found as follows, without the intervention of series.)

7. *To find the differential of the product of two variables*, say xy.

$\because\ 2xy=(x+y)^2-x^2-y^2.$

$\therefore\ 2d(xy)=2(x+y)\times d(x+y)-d(x^2)-d(y^2).$

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*That is, constant during an infinitely small interval.

$$2d(xy)=2(x+y)(dx+dy)-2xdx-2ydy.$$

i. e. $d(xy)=xdx+xdy+ydx+ydy-xdx-ydy.$

i. e. $d(xy)=xdy+ydx.$

From this may be derived rules for $d(xyz)$, etc., and $d(x/y)$.

8. Here would follow demonstration by differentials of Binomial Formula for all values of n, with exercises.

9. Here would follow the algebraic deduction of some such formula as:

$$\log(1+z)=M(z-\tfrac{1}{2}z^2+\tfrac{1}{3}z^3-\ldots\ldots \text{ ad inf.})$$

$$\text{whence } d(\log 1+z)=M(1--z+z^2-\ldots\ldots\text{ad inf.})dx$$

$$d(\log 1+z)=M.\frac{1}{1+z}.dx$$

or, putting x for $1+z$ we have

$$d\log x=M(dx/x).$$

Whence may be derived $d(a^x)=a^x\log a$, etc.

10. Here would follow the algebraic deduction of

$$\sin x=x-(x^3/3\,!)+(x^5/5\,!)-(x^7/7\,!)+\ \ldots\ldots \tag{1}$$

$$\text{and } \cos x=1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots \tag{2}$$

From (1), $d(\sin x)=\{1-(x^2/2\,!)+(x^4/4\,!)-(x^6/6\,!)+\ldots\ldots\}dx=\cos x dx$,

and from (2), $d(\cos x)=-\sin x dx$, etc.

11. Then might follow applications to questions of maxima and minima, etc. Then deduction of Taylor's Theorem, with applications.

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ARITHMETIC.

Conducted by B. F. FINKEL, Springfield, Mo. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

79. Proposed by F. M. PRIEST, St. Louis, Mo.

How many $20 gold pieces can be put in a room 20 feet long, 18 feet wide, 9 feet high?